MATHEMATICAL VISUALIZATION

A PROJECT OF THE NORTHERN NEW MEXICO COLLEGE, VISUAL COMMUNICATION PROGRAM

MULTIMEDIA PRESENTATION HAND OUT

The following is an attempt to revisit a significant tradition applied to the teaching of mathematics and encourage students to develop further their interest and intellectual curiosity in the field of mathematics and mathematical visualization.

The concept of this presentation is based on the research of Dr T. Rothman & F. Hidetoshi on Japanese Temple Sangaku tablets, an original expression of Mathematical Visualization that flourished in 18th and 19th century Japan.

Students developed their work from the original templates and problem solving description of Alexander Bogomolny.

Jean Constant
acting director,
NNMC, Visual Communication Program

1- Tyra Portis
"Two Arbelos - Two chains"
Sangaku written in 1842 in the Nagano prefecture

Problem

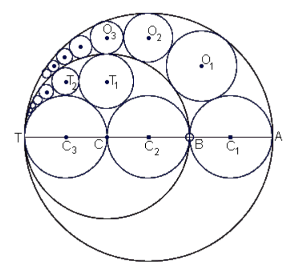
- Points T, A, B, C are collinear and AB = BC = CT = 2r.
- Circles $S_1(3r)$ and $S_2(2r)$ are drawn on AT and BT respectively as diameters.

The chain of contact circles $O_i(r_i)$, i = 1, 2, ...,where $O_1(r_1)$
- touches $C_1(r)$, drawn on AB as diameter,
- touches $S_1(3r)$ internally and $S_2(2r)$ externally, and so on.

The circles $C_2(r)$ and $C_3(r)$ with respective diameters BC and CT construct another chain of contact circles $T_i(t_i)$, i = 1, 2, ...

Prove that:

$$\mathbf{t_n / (t_n / r_n - 3) = 2r / 13}$$



Solution

- Small arbelos: $t_n = 2r / (n^2 + 2)$
- Big arbelos: $r_n = 3r\cdot 1/2 / ((n/2)^2 + 1/2 + 1) = 6r / (n^2 + 6)$

In order to derive at anything resembling 2r/13, we need to remove the dependency on n. Passing to the reciprocals:

$$1 / t_n = (n^2 + 2) / 2r \text{ and}$$
$$1 / r_n = (n^2 + 6) / 6r$$

We obtain

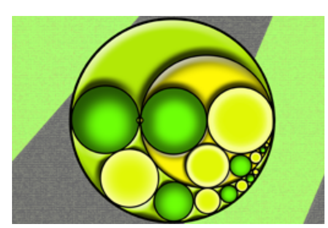
$$3 / r_n - 1 / t_n = 4 / 2r = 2 / r, \text{ independent of n.}$$

The latter can be rewritten as

$$1 / r_n \cdot (3 - r_n / t_n) = 2 / r,$$

or:

$$\mathbf{r_n / (3 - r_n / t_n) = r / 2}$$

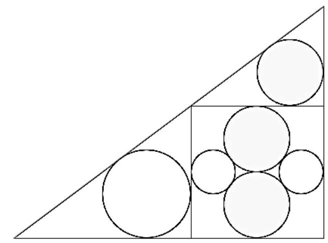


2- Steve Wold
“3-4-5 Triangle”
By Sato Naosue, 1847

Problem

- Two circles of radius r and two circles of radius t are inscribed in a square.
- The square itself is inscribed in a large triangle and,
- Two circles of radii R and r are inscribed in the small triangles outside the square.

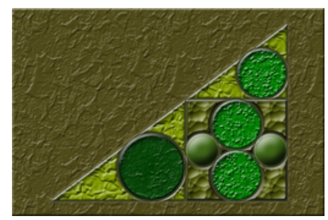
Show that:
R = 2t



Solution

- The side of the square is of length 4r.
- r = 3t/2.
- The small upper triangle = 3-4-5 with the short side is equal to 3r.
- From the similarity of the three triangles, all of them have the proportions 3-4-5 which leads to 4r = 3R.

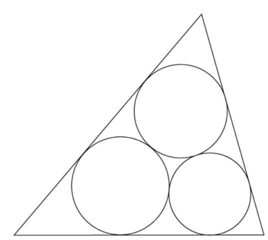
And finally,
R = 4r/3 = 2t.



3- Chris Trujillo
“Malfatti's problem”
By Chokuyen Naonobu Ajima, 1732-1798

Problem

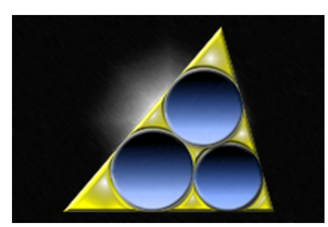
Cut out of a triangular prism three circular columns of the greatest possible volume



Solution

1. Draw three angle bisectors IA, IB, IC.
2. In the triangles IAB, IBC, ICA inscribe circles Cc, Ca, Cb. Note that the angle bisectors serve as common internal tangents for pairs of these circles.
3. For each pair of the circles consider the second internal tangents. The latter concur in a point (L) and cross the sides in points M, P, T, as shown in the applet.
4. The three quadrilaterals APLT, BMLP, and CTLM are inscriptible.

Their incircles solve Malfatti's problem.

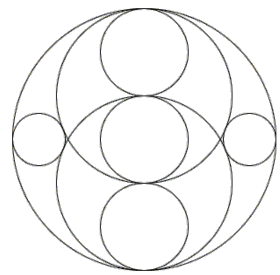


4- **Heather Martinez**
"Sangaku with 8 Circles"
From the IchiNoSeki museum

Problem

As out of eight circles six have obvious relations between their radii

Find the radius of the two small circles in terms of the radius of the bigger one.



Solution

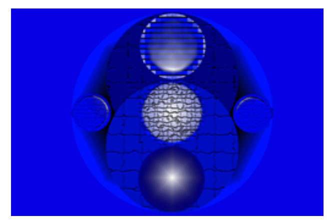
Assume that:
- $3r$ is the radius of each circle in the vertical triplet and,
- $2r$ is the radius of the two bigger twins.
- x is the unknown radius of the small circles.

In ΔABO, $AB = 2r + x$,
$OB = r$,
$OA = 3r - x$.

$r^2 + (3r - x)^2 = (2r + x)^2$ and $6r^2 = 10rx$ and $x = 3r/5$ $6r^2 = 10rx$, or $x = 3r/5$

If the radius of the big circle is $R = 3r$,

$$\mathbf{x = R/5}$$

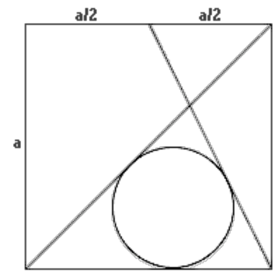


5- **Jeramy Martinez**
"Sangaku in a Square"
Unknown origin

Problem

A triangle is formed by a line that joins the base of a square with the midpoint of the opposite side and a diagonal.

Find the radius of the inscribed circle.



Solution

In ΔBCQ.

- MD is parallel to the base and is half as long which implies that this is a midline of the triangle.
- M and D are the midpoints of BQ and CQ, respectively.
- BD and CM are two medians in the triangle and
- P its centroid.

The centroid divides the medians in ratio 2:1 so that

- $CP = CM\cdot 2/3$
- $BP = BD\cdot 2/3$.

Assuming $BC = a$. In any triangle, $r\cdot p = 2S$

r is the inradius,
p the perimeter, and
S the area of the triangle.

$r\cdot a\cdot(1 + \sqrt{5}/2\cdot 2/3 + \sqrt{2}\cdot 2/3) = 2\cdot a2/3$. From which:

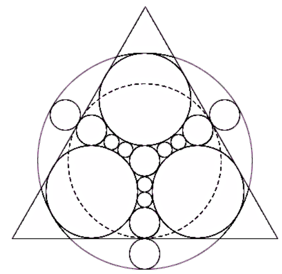
$$\mathbf{r = 2\cdot a / (3 + \sqrt{5} + 2\sqrt{2})}$$

6- **Mario Delgadillo**
"Simple Sangaku"
By Tanabe Shigetoshi, 1865

Problem

In an equilateral triangle,
- three circles of radius a
- four circles of radius b
- six circles of radius c

touch each other as shown. If
R is the radius of the outer circle and,
r is the radius of the dashed circles,
find c in terms of r

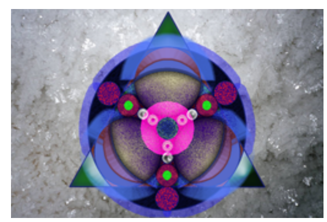


Solution

$r = 3b + 4c,$
$R = 5b + 4c,$
$R = b + 2a,$
$a + b = 2b + 4c$

Solving these simultaneously we see that

$$b = 2c$$
$$a = 6c$$
$$\mathbf{r = 10c}$$



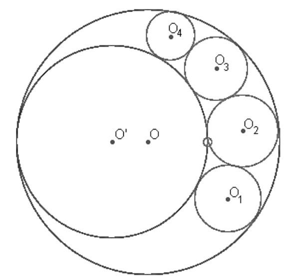
6- Josh Romero
"1+27 = 12+16"
Circa 1810. From the Fukagawa and Pedoe collection

Problem

The circle O'(r') touches O(r) internally and a chain of contact circles $O_i(r_i)$, i = 1, 2, 3, 4, is inscribed in the lune formed by O(r) and O'(r').

Show that:

$$1 / r_1 + 3 / r_3 = 3 / r_2 + 1 / r_4$$



Solution

The problem is solved by a general formula for the radius of the circles in the lune:

$$r_t = r\, r'(r - r') / (r\, r' + t^2(r - r')^2)$$

where circles in the chain tangent to each other correspond to values of t different by 1

Using A = r r' and B = r - r', we can write:

$r_1 = A / (A + B^2)$,

$r_2 = A / (A + 2^2B^2)$,

$r_3 = A / (A + 3^2B^2)$,

$r_4 = A / (A + 4^2B^2)$.

Substituting this into:

$$1 / r_1 + 3 / r_3 = 3 / r_2 + 1 / r_4.$$

we can verify that

$$\mathbf{1 + 3 \cdot 3^2 = 3 \cdot 2^2 + 4^2}$$